Mathematics for Human Flourishing

By Francis Su with
reflections by
Christopher Jackson

Contents

Simone Weil, around 1937.
Photo courtesy of Sylvie Weil.

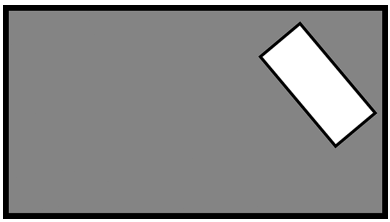
A meeting of Bourbaki, around 1938. Simone Weil is seen at left, leaning over her notes. André Weil is waving the bell. Photo courtesy of Sylvie Weil.

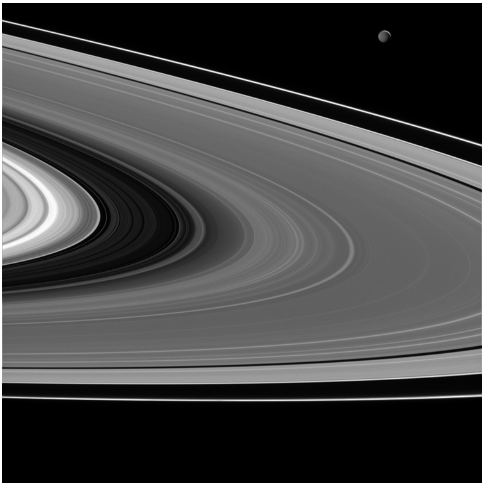
DIVIDING BROWNIES

A father bakes brownies in a rectangular pan as an after-school snack for his two daughters. Before his daughters get home, his wife comes along and removes a rectangle from somewhere in the middle, with the sides of the rectangle not necessarily parallel to the sides of the pan.

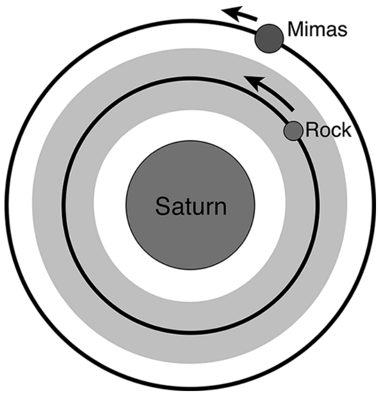
How can he make one straight cut and divide the remainder of the brownies evenly between his two daughters so that they get the same area?

A version of this puzzle was featured on the NPR show *Car Talk.*[a]

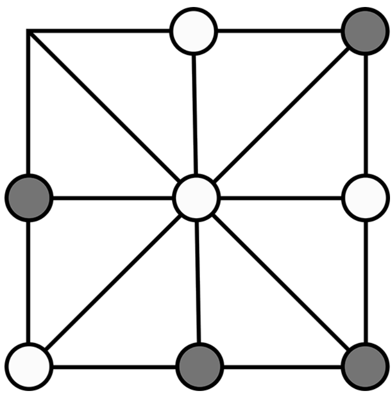




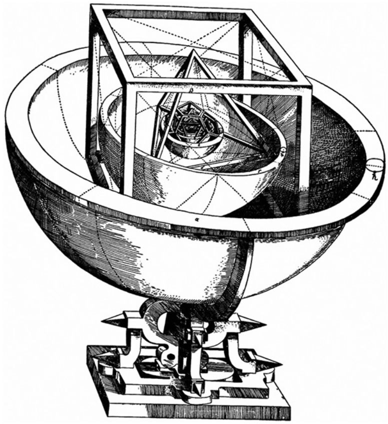
Mimas by Saturnshine. The moon Mimas is being illuminated by sunlight reflected off Saturn. The Cassini division is the largest gap in the rings, visible on the left side of the photo. Image courtesy NASA/JPL-Caltech/Space Science Institute. Taken by the *Cassini* spacecraft on February 16, 2015.



Icy rock in an inner orbit around Saturn catching up with Mimas. The effects of Mimas's gravity may accumulate and perturb the rock's orbit if the rock always passes the moon in the same location.

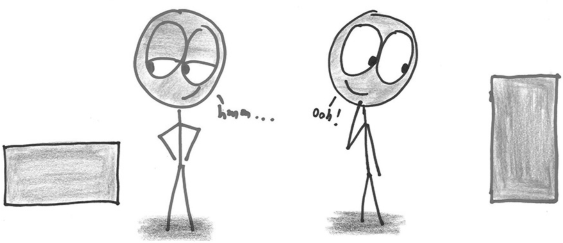


All eight pieces have been placed on this Achi board and there is no winner yet, so the players will take turns pushing one of their pieces onto the empty position until one gets three in a row.

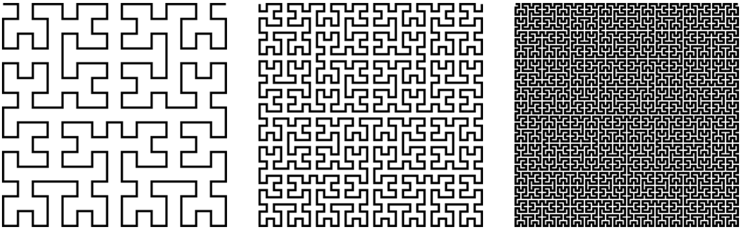


Kepler's model of the solar system in
Mysterium Cosmographicum.

Create two rectangles so that the first has a bigger perimeter, and the second a bigger area.

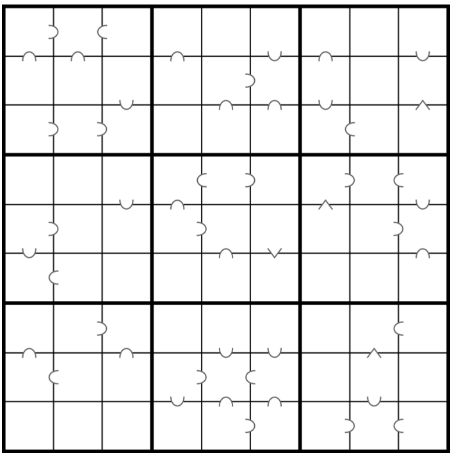


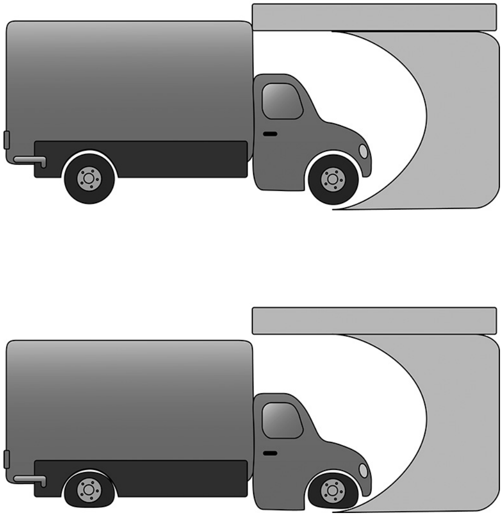
A Bad Drawing, courtesy of Ben Orlin, from his
book *Math with Bad Drawings.*



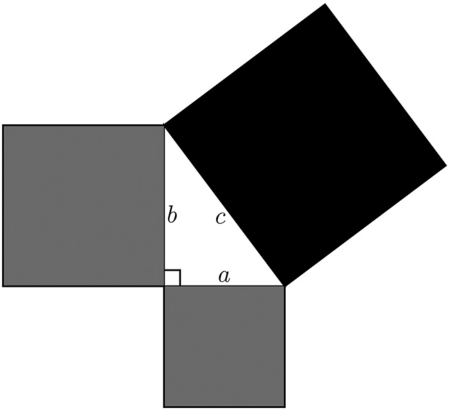
A space-filling curve is constructed as a limit of curves like these, which wind their way through a given region of space in a denser wayeach time. The mathematical work here is to show that such a limit exists.

SUDOKU Board

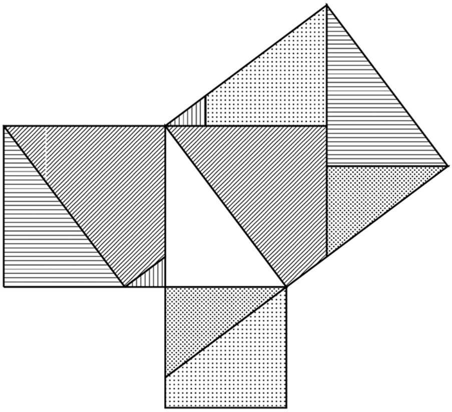




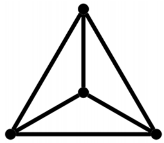
Two Trucks



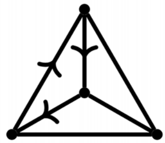
Geometric Story



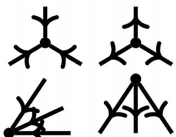
Explanatory Story



starting diagram



a cycle cell



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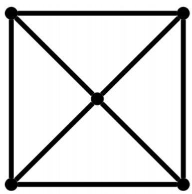

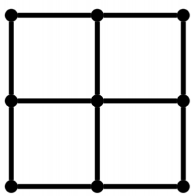

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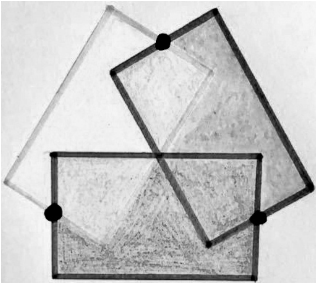
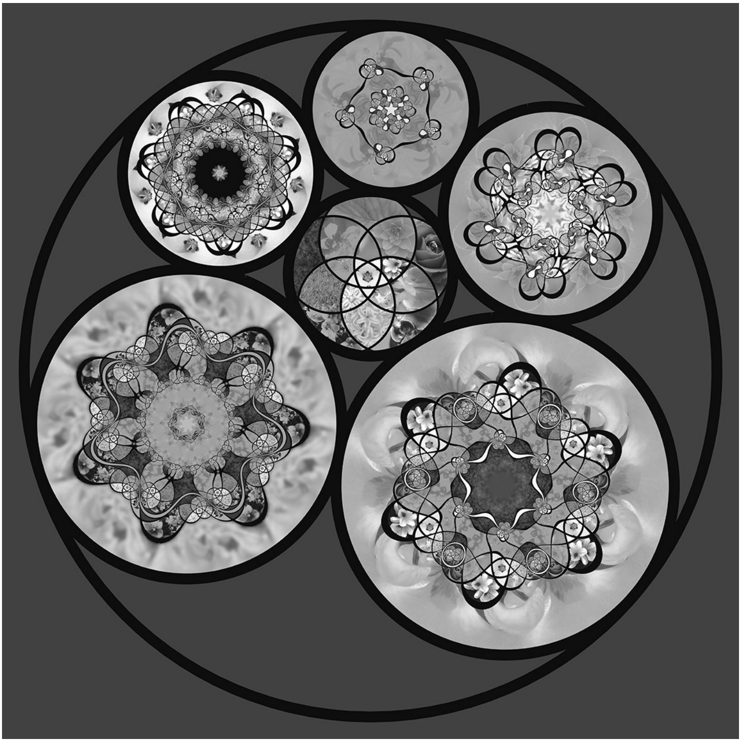
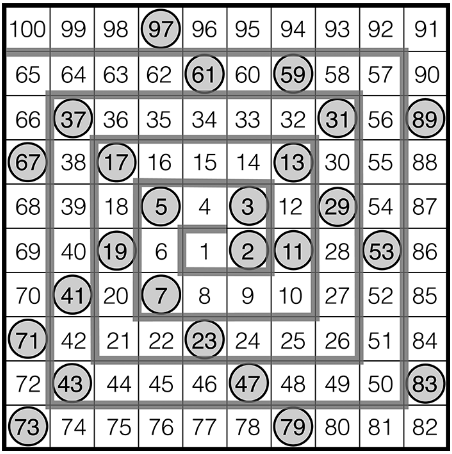




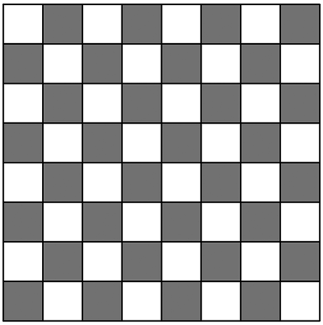
Floral rosettes by the mathematician Frank Farris, who uses techniques from complex analysis to create art from photographic material. This set illustrates Steiner's porism, a theorem about when circles can be inscribed in the area between two given circles in a way that completes a ring, like the one shown here. The outer circles are filled with rosette patterns whose colors come from the central floral collage (the original version is in color, but even in black and white this artwork is striking).

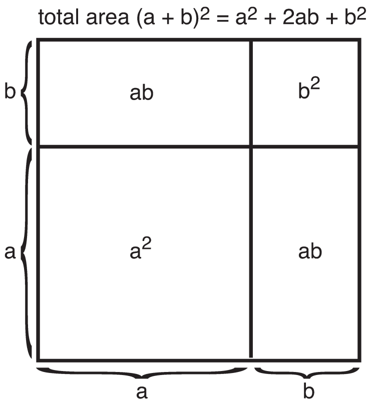


The Ulam spiral. Primes (circled) seem to crowd in diagonal straight lines.

The Sydney Opera House.

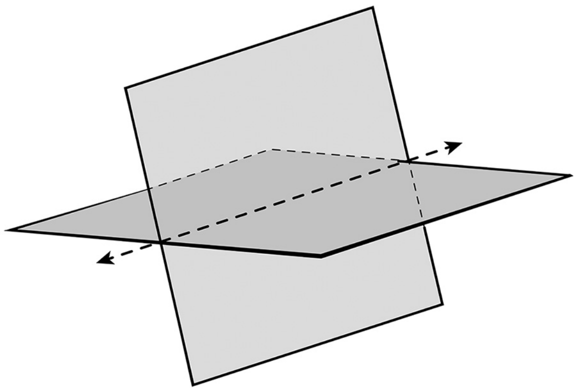
CHESSBOARD





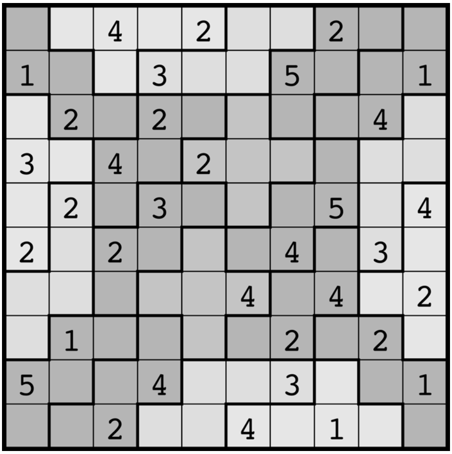
This was not part of Smith's address, but I can't resist showing you a "proof without words" that $(a + b)^2 = a^2 + 2ab + b^2$.

Sliding block puzzle, made by Kametaro Matsumoto, in its starting configuration (left) and in an ending configuration (right). Photos courtesy of Shinya Ichikawa; sliding block puzzle courtesy of Jean Matsumoto and Alice Ando.

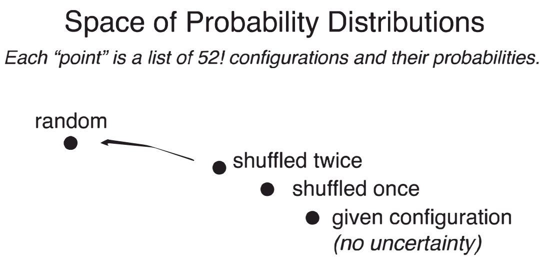


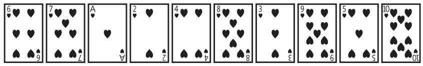
Two planes intersecting.

PENTOMINO SUDOKU

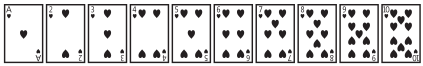


a. Philip Riley and Laura Taalman, Brainfreeze Puzzles, *Double Trouble Sudoku* (New York: Puzzlewright, 2014), 189.

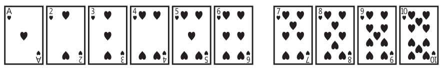




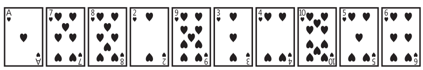
Ten Card Sequence 1



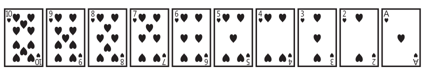
Ten Card Sequence 2



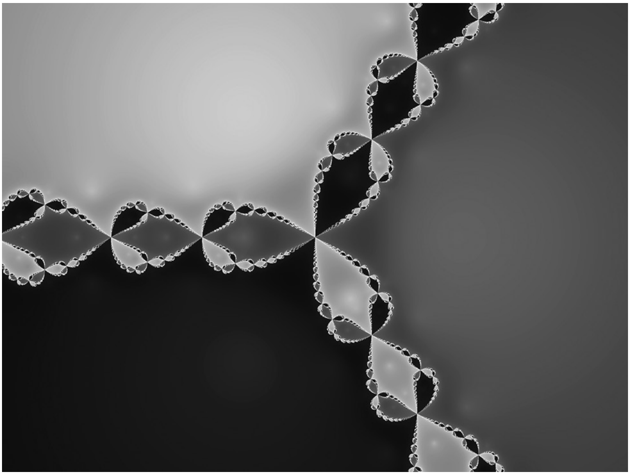
Ten Card Sequence 3



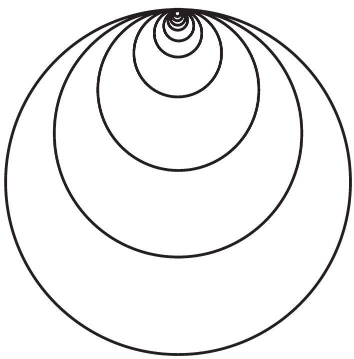
Ten Card Sequence 4



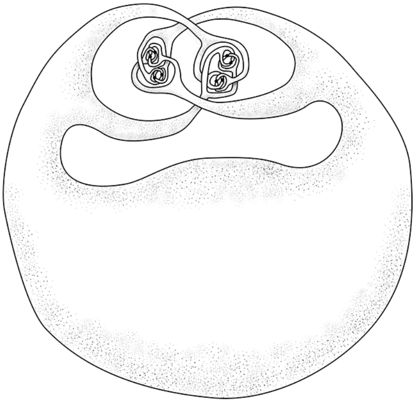
Ten Card Sequence 5



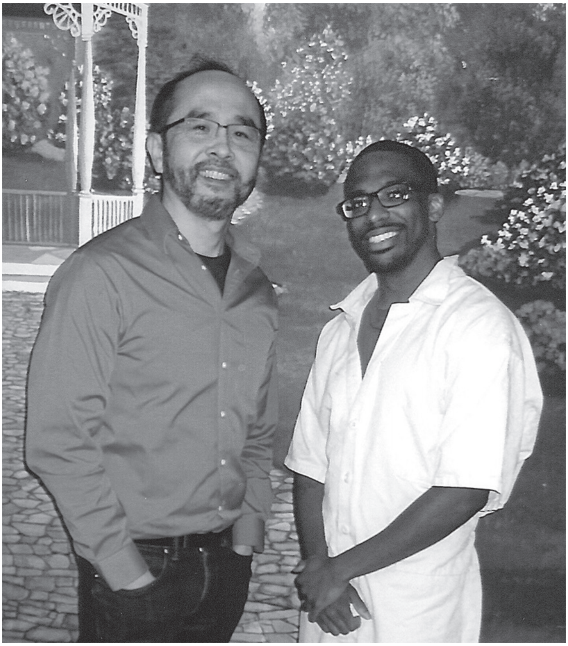
This fractal picture has three regions (dark-, medium-, and light-shaded "lakes") that share the same boundary, though unlike the original Lakes of Wada, each lake here consists of disconnected basins.



A Hawaiian earring.



The Alexander Horned Sphere.



After five years of correspondence from opposite sides of the country, Christopher Jackson and I met in person for the first time in November 2018. The background is a mural painted on a prison wall, the only place where prison administration would allow a photo.

desires & virtues

Here is a list of all the virtues mentioned in this book, which are built through the pursuit of mathematics when it is grounded in basic human desires. The desires are the chapter titles, and the virtues I chose to discuss are listed under each.

EXPLORATION
imagination
creativity
expectation of enchantment

MEANING
story building
thinking abstractly
persistence
contemplation

PLAY
hopefulness
curiosity
concentration
confidence in struggle
patience
perseverance
ability to change perspectives
openness of spirit

BEAUTY

reflection
joyful gratitude
transcendent awe
habits of generalization
disposition toward beauty

PERMANENCE

trust in reason

TRUTH

thirst for deep knowledge
thirst for deep investigation
thinking for oneself
thinking rigorously
circumspection
intellectual humility
admitting error
confidence in truth

STRUGGLE

endurance
unflappable character
competence to solve new problems
self-confidence
mastery

POWER

skill in interpretation, definition, quantification, abstraction, visualization, imagination, creation, strategization, modeling, multiple representations, generalization, and structure identification
humble character
sacrificial character
encouraging character

heart of service
resolve to unleash creativity in others
resolve to elevate human dignity

JUSTICE

empathy for the marginalized
concern for the oppressed
willingness to challenge the status quo

FREEDOM

resourcefulness
fearlessness in asking questions
independent thinking
seeing setbacks as springboards
confidence in knowledge
inventiveness
joyfulness

COMMUNITY

hospitality
excellence in teaching
excellence in mentoring
disposition to affirm others
self-reflection
attention to people
vulnerability

LOVE

love, the source of and end of all other virtues

for reflection
questions for further discussion

Our views and our practices of mathematics cannot change without reflection and action. I am providing some questions below as starting points for further discussion. On my webpage (francissu.com), I am maintaining other resources associated with the book that may be helpful to teachers, including reading lists with live links to references.

FLOURISHING

These first three questions are ones you may wish to think about before reading the book, and then return to after you finish it, to see how your answers compare.

1. What is mathematics? How would you describe it to a friend, in a sentence or two? What do you feel is the purpose of learning mathematics, for yourself or others?
2. What connections do you see between doing mathematics and being human?
3. Describe any virtues you have acquired as a result of doing mathematics.

EXPLORATION

1. Think of a time when you were captivated by exploring something (e.g., a location, an idea, a game). What analogies can you draw between doing math and doing this exploration?
2. Consider this statement: "The wayfinders were mathematical ex-

plorers of their society, using attentive study, logical reasoning, and spatial intuition to solve the problems they encountered in their cultural moment." Choose any cultural practice and reflect on ways that mathematical thinking might present itself in that practice.

3. If you teach math to others, what are some ways that you can train your students to expect enchantment?

MEANING

1. "Mathematical ideas are metaphors." Reflect on one mathematical idea that you've now seen in multiple situations, and how the meaning of that idea was enhanced in each encounter.
2. How does abstraction enrich the meaning of an idea? Describe one example from your own experience.
3. "Mathematics is the art of engaging the meaning of patterns." Consider this statement in light of a scientific discovery in which mathematics played a part.

PLAY

1. Think of an activity you associate with play. Make a list of all the things you enjoy about its playful aspects. Does your list have analogies in mathematical activities?
2. Some people seem to have patience and hopefulness when trying to solve a problem, and they will persist for a long time in thinking about it. Others seem to give up quickly. How does math play build hopefulness and patience? Compare this to the discipline of learning a sport.
3. Math play "asks you to change perspective, to look at a problem from different viewpoints." In what ways is this virtue useful in life?

BEAUTY

1. Describe any experience you've had with sensory, wondrous, insightful, or transcendent mathematical beauty. How did that experience make you feel?
2. Think about all your educational experiences—for instance, classes

you've taken in different subjects. Which ones implicitly acknowledged the human desire for beauty?
3. Where is mathematical beauty found in the world?

PERMANENCE

1. What mathematical laws, truths, or ideas do you rely on in your daily life?
2. How is mathematics a refuge? For whom is it a refuge?
3. Many things in the universe change over time (and let's not forget—the subject of calculus was developed to study such change). Do you find it surprising that the laws of mathematics do not change over time?

TRUTH

1. Reflect on a time when shallow knowledge (in any subject) has led you astray. How did that make you feel? How is deep knowledge an antidote?
2. Sometimes parties on two sides of an argument have different perspectives on the same event. Both views may be true, but each may be just part of the picture. How is knowing the whole truth a better place to be? Similarly, in mathematics, what does knowing the whole truth look like?
3. How can mathematical thinking equip you to converse with and respect people who hold different views?

STRUGGLE

1. Describe an activity you enjoy, and make a list of all the internal and external goods you can think of that are associated with that activity. Now think of an activity you don't enjoy, and make a similar list. What do you notice about these lists?
2. What internal goods does mathematics offer? Discuss how these goods multiply when you share them with others.
3. If you teach mathematics, how can you incentivize students to value the process of struggle and not just the outcome?

POWER

1. Think of a recent challenging math problem you explored. Which of the powers of mathematics did you develop or use in that exploration (interpretation, definition, quantification, abstraction, visualization, imagination, creation, strategization, modeling, multiple representations, generalization, structure identification)?
2. Discuss creative power and coercive power that you have witnessed in mathematical settings.
3. If you teach, how do you affirm your students' dignity as creative human beings in the way that they do mathematics?

JUSTICE

1. If people have realized that the way we teach math needs to change, why hasn't it changed yet? Who benefits from keeping it the same way it has always been?
2. All of us unwittingly harbor bias, so how can we mitigate bias in mathematical spaces? Who is harmed by bias in mathematical spaces, and why?
3. What inequities do you notice in mathematical spaces? Who is harmed by those inequities? Think deeper than the obvious answers.

FREEDOM

1. Describe settings in which you've experienced any of these freedoms: the freedom of knowledge, the freedom to explore, the freedom of understanding, or the freedom to imagine.
2. Who do you think may not be feeling welcome in mathematical spaces? In what ways can you extend the freedom of welcome in mathematics to those around you? Think of concrete actions you can take.
3. What things have you experienced in a math classroom that feel like freedom? What things feel like domination?

COMMUNITY

1. Why should hospitality, or excellence in teaching and mentoring, be central to doing mathematics well?
2. How can you build a community in the classroom or the home in which participants push one another to grow while not being overly focused on achievement?
3. What actions can you take to address feelings of not belonging in math communities?

LOVE

1. In what harmful ways do we use mathematics as "a showcase for flaunting talent rather than a playground for building virtue"?
2. How can you honor each person you meet as a dignified mathematical thinker?
3. Who are the forgotten among you, mathematically speaking? Whom will you love, whom will you read differently?

hints & solutions to puzzles

Before looking at any of these hints or solutions, you should first play with the puzzles! Work through examples to get a feeling for what is going on. Take as much time as you want to dwell on the problems—there is no hurry. The struggle itself is valuable.

HINTS

Dividing Brownies. Try special cases. If the removed rectangle is very tiny, how should you orient the cut?

Toggling Light Switches. Try several examples. Look at particular bulbs, and ask which multiples will toggle them.

"Divides" Sudoku. Since within each three-by-three block all pairs of adjacent cells that are related by division are marked with the ⊂ symbol, you can identify the locations of most of the 1s. After that, start looking for chains: for example, if you see A ⊂ B ⊂ C and you know that none of A, B, or C is a 1, then the only possibility is 2 ⊂ 4 ⊂ 8. Also look for cells that divide more than one neighbor or are divisible by more than one neighbor. Notice that 5 and 7 do not divide any other number from 1 through 9 and are not divisible by any other number from 2 through 9.

Red-Black Card Trick. If the black cards from the second pile were replaced by the red cards from the first pile, would the size of the second pile change?

Water and Wine. How is this puzzle like the Red-Black Card Trick?

The Game of Cycles. Explore the game and make some conjectures. It may help to notice: if a triangular cell has exactly one edge marked with an arrow, and you mark a second edge with an arrow pointing in the same direction as the first arrow (so as to partially form a cycle), then the other player can win on the very next move by completing the cycle.

A Geometric Puzzle. Look at the area of the regions where two rectangles overlap. What can you say about this area? Can you cut this region into pieces whose areas are easy to figure out?

Ants on a Log. Suppose you had only two ants instead of one hundred. What can you say about the configuration of ants before and after a collision?

Chessboard Problems. Each domino covers one black square and one white square. If one black square and one white square are removed from the board, must it always be possible to cover it with dominoes? Where does a knight sitting on a white square go after one move? What color squares does a Tetris piece cover? How can you color the little squares in an $8 \times 8 \times 8$ cube so that $1 \times 1 \times 3$ blocks cover the same number of each color type?

Matsumoto Sliding Block Puzzle. You can construct the sliding block puzzle out of cardboard or paper. How will you get the large square piece past the horizontal rectangular piece?

Shoelace Clock. The answer to part 1 is less than 7.5 minutes. The answer to part 2 is surprising.

Vickrey Auction. Try to show that a bidder who bids her true value V will never do worse and will sometimes do better than bidding some other value B.

Pentomino Sudoku. Start by looking for double 2s or double 4s in the rows and columns. What does this say, for instance, about the pentomino in the bottom left corner? Counting the number of particular digits required in neighboring rows or columns can be a useful tool.

Power Indices. The set of all orderings of three groups is *ABC, ACB, BAC, BCA, CAB, CBA*. In which of these is group *C* pivotal?

Unknown Polynomial. You need a lot fewer questions that you might first suspect. The fact that the coefficients are nonnegative is important here. Can you bound the size of the largest coefficient?

Five Points on a Sphere. Remember, your goal is to show that no matter what five points on a sphere you pick, there's a hemisphere that contains four of them. You might be tempted to guess the "worst" configuration and show that the claim holds for it, but that is not sufficient to show that the claim holds for *every* configuration. Also, for any pair of points, is there an easy way to ensure that the pair lies inside a hemisphere?

SOLUTIONS

Dividing Brownies. If you cut along a straight line that runs through the center of the large rectangular pan and the center of the rectangular hole, the cake pieces on both sides will have the same area, since each will be half the size of the pan minus half the size of the hole.

Toggling Light Switches. The bulbs that are on at the end of all the toggling are those numbered 1, 4, 9, 16, 25, 36, 49, 64, 81, and 100. These are the bulbs corresponding to perfect squares. To see why, notice that bulb N gets toggled for each switch whose number is a factor of N (a number that perfectly divides N). Perfect squares are the only integers with an odd number of factors. You can see this because most factors come in pairs: if J is a factor of N, then N/J is also a factor of N, and the only instance where J and N/J are not distinct is when $J = N/J$. In that case $J^2 = N$, so N is a perfect square.

"Divides" Sudoku. See below.

3	1	7	2	5	8	4	9	6
9	5	6	4	3	1	8	7	2
8	4	2	7	6	9	1	3	5
5	7	8	1	9	3	6	2	4
6	3	4	8	2	7	9	5	1
1	2	9	6	4	5	7	8	3
2	9	1	5	7	6	3	4	8
4	8	3	9	1	2	5	6	7
7	6	5	3	8	4	2	1	9

Red-Black Card Trick. The reason this trick works can be seen in a couple of ways. Let H be half the number of cards in the deck. If the number of red cards in the first and second piles are respectively R and S, and the number of black cards in the first and second piles are respectively A and B, then we know that $R + S = H$ (since the total number of red cards is H) and $S + B = H$ (since the total number of cards in the second pile is H). Then both R and B equal $H - S$, so $R = B$. Another way to see it is this: if you move the R red cards to the second pile and remove the B black cards from the second pile, the second pile now has all the red cards and is the same size (H) as before. So R must equal B.

Water and Wine. Let H be the total volume of water. If, after this process, the amount of water in the wine glass is R and the amount of wine in the water glass is B, then replacing B by R would leave the volume H unchanged. So R and B must be equal.

The Game of Cycles. The second player has a winning strategy. After the first player begins by marking an edge with an arrow, the second player should respond by marking the only edge in the diagram that does not touch the first edge (the direction of the arrow doesn't matter). After that, as long as the second player doesn't complete the second edge of a potential cycle cell, she will win. There are many other good questions to explore. For other starting diagrams, who has a winning strategy? Are there starting diagrams for which it is possible to mark every edge with an arrow and have no cycle cells?

A Geometric Puzzle. Each pair of rectangles overlaps in a quadrilateral Q. Cut Q along a line from P, the point where the three rectangle borders meet, to the point M, the other point where the two rectangle borders meet. This splits the overlapping region into two triangles that have the same area (by symmetry) and are each corners of a rectangle. It can now be seen that those corner triangles are 1/8 of the total area (4) of a rectangle. So the area of an overlapping region is 1, and there are three overlapping regions. So the total area covered by all three rectangles is 3 times 4 minus 3, which is 9.

Ants on a Log. While ants bouncing off each other seem difficult to keep track of, one key idea makes it quite simple: two ants bouncing off each other is *equivalent* to two ants that pass through each other, in the sense that the positions of the ants in each case are identical. So you might as well think of all the ants as acting with independent motions. From this viewpoint, the longest that you would need to wait to ensure that all the ants were off is the amount of time needed for a single ant to traverse the length of the log, which is 1 minute.

Chessboard Problems. Chapter 6 contains a proof that when two squares of the same color are removed from a chessboard, what remains cannot be tiled by dominoes. If two squares of opposite colors are removed, the remainder *can* be tiled by dominoes—to see this, find a continuous path, moving from square to adjacent square, that visits every square on an 8×8 board. Removing one black and one white square breaks this path into two paths, each of which has an even number of squares, and each path can thus be tiled by dominoes.

For knights on a 7×7 chessboard, notice that in each move, a knight lands on a square of the opposite color from where it started, so simultaneous legal moves would be possible only for a board with the same number of black and white squares. But a 7×7 chessboard does not have an equal number of black and white squares.

For the Tetris question, remember that the seven Tetris quadrominoes correspond to the shapes of the letters *O, I, L, J, T, S,* and *Z.* Notice that all the Tetris pieces cover the same number of black and white squares except for one—the T-shaped piece—so covering a 4×7 chessboard with them is impossible.

For the 8×8×8 cube with opposite corners removed, using coordinates to specify the location of the 1×1×1 cubes, color the 1×1×1 cube at position (i, j, k) with one of three colors, corresponding to the remainder of $i + j + k$ when you divide it by 3. Since a 1×1×3 block will then cover exactly one 1×1×1 cube of each color, the figure can be tiled by 1×1×3 blocks only if there are the same number of cubes of each color in the figure. But there are not.

Matsumoto Sliding Block Puzzle. If the tiles in the starting configuration are numbered as 2 for the young woman; 1, 3, 4, and 6 for the vertical dominoes (numbered left to right and top to bottom); 5 for the horizontal domino; and 7, 8, 9, and 10 for the small squares (numbered left to right and top to bottom), then a solution from the starting configuration shown goes like this: 6, 10, 8, 5, 6, 10 (halfway), 8, 6, 5, 7 (up, left), 9, 6, 10 (left, down), 5, 9, 7, 4, 6, 10, 8, 5, 7 (down, right), 6, 4, 1, 2, 3, 9, 7, 6, 3, 2, 1, 4, 8, 10 (right, up), 5, 3, 6, 8, 2, 9, 7 (up, left), 8, 6, 3, 10 (right, down), 2, 9 (down, right), 1, 4, 2, 9, 7 (halfway), 8, 6, 3, 10, 9 (down), 2, 4, 1, 8, 7, 6, 3, 2, 7, 8, 1, 4, 7 (left, up), 5, 9, 10, 2, 8, 7, 5, 10 (up, left), 2.

Shoelace Clock. 1. There is a method for measuring 3.75 minutes. Suppose the shoelace has endpoints *A* and *B*. Since it is symmetrical, a cut at its midpoint will produce two identical laces that are each possibly nonsymmetrical and have a burn time of 30 minutes. Lay them side by side such that *A* and *B* are side by side. Burn both ends of one lace. After 15 minutes, where they meet and burn out, cut the other lace at the corresponding point. Now you have two laces with no relationship with each other except that they both have a burn time of 15 minutes. Simultaneously light both ends of one lace and one end of the other. When the flames on the first lace meet, after 7.5 minutes, extinguish the flame on the other lace. What remains is a lace that will burn in 7.5 minutes. Lighting both ends will measure 3.75 minutes, when the flames meet.

2. You can measure arbitrarily short intervals of time! For instance, you can measure any time interval that is 60 minutes divided by a power of 2. To see this, cut the identical, symmetric shoelaces at their midpoints to produce four identical nonsymmetric shoelaces. Ignore one, and you have three laces.

We will outline a procedure that will take three laces, two identical, of equal burn time *T* and produce a set of three laces, two identical, of burn time *T*/2. Call the laces lace 1, lace 2, and lace 3, each with burn time *T*, where lace 1 and lace 2 are identical and laid side by side in that manner. Simultaneously light both ends of lace 3 and one end of lace 2. When the flames on lace 3 meet and burn out, extinguish the flame on lace 2 and cut lace 1 at the corresponding position. You now have three laces with burn time $T/2$, two of which are identical.

You can continue this process indefinitely to produce laces with a burn time of the form $T/2^k$.

Vickrey Auction. Here's why a bidder's best strategy is to bid V, what she thinks the car is worth. Let M be the (unknown) maximum of all other bids. No matter what M is, we'll show that bidding any other amount B is never better than bidding V. If both V and B are less than M, the bidder loses the car in either instance, and if both V and B are greater than M, the bidder wins the car and pays M in either instance. So the only scenario where there's a difference in outcome between bidding B and bidding V is if M lies between B and V.

If $B > M > V$, then bidding B is bad for the bidder because she will win the auction but will pay M, more than she feels the car is worth, hence losing money on that transaction; whereas if she had bid V, she would have lost the auction and had no net change to her fortune.

If $B < M < V$, then bidding B is bad for the bidder because she'll lose the auction and the net change to her fortune will be zero, whereas if she had bid her true value V, she would have won the auction and paid M, less than she thought the car was worth, netting a gain to her fortune.

Pentomino Sudoku. See below.

5	3	4	1	2	5	1	2	4	3
1	4	5	3	4	2	5	3	2	1
1	2	3	2	3	5	1	5	4	4
3	5	4	5	2	1	4	3	1	2
4	2	1	3	1	3	2	5	5	4
2	5	2	4	5	1	4	1	3	3
3	1	5	2	3	4	5	4	1	2
4	1	3	5	1	3	2	4	2	5
5	4	1	4	5	2	3	2	3	1
2	3	2	1	4	4	3	1	5	5

Power Indices. The six orderings of three groups are *ABC, ACB, BAC, BCA, CAB, CBA*. If group *A* is size 48, *B* is size 49, and *C* is size 3, then the middle group in each ordering is pivotal. So the power of each group, according to the Shapley-Shubik index, is 1/3.

Unknown Polynomial. You need only two questions to determine the polynomial. First ask for the value of the polynomial at 1. The answer will give you the sum of the coefficients, which must be larger than every coefficient if they are all nonnegative. If the number of digits in the answer is *k,* then ask for the value of the polynomial at 10^{k+1}. The digits of the answer will display the coefficients of each term within blocks of size $k + 1$. For example, if the value of the polynomial at 1 is 1044, then the largest coefficient is not more than four digits long. Now ask for the value of the polynomial at 10^5. If the answer is 12003450067800009, then counting from the right, each block of size 5 displays a coefficient of the polynomial, which must therefore be $12x^3 + 345x^2 + 678x + 9$.

Five Points on a Sphere. Choose any pair of points from our collection of five points. These points determine a great circle that divides the sphere into two hemispheres (a great circle on the sphere is a circle whose center is the center of the sphere). The two points thus lie on the boundary of both hemispheres. Of the three other points in our collection, at least two must be contained in one of these hemispheres. So that hemisphere contains those two points plus both of the initial pair of points—all together, that's four points.